

Diffraction by an impedance strip I. Reducing diffraction problem to Riemann–Hilbert problems

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Abstract

A 2D problem of acoustic wave scattering by a segment bearing impedance boundary conditions is considered. In the current paper (the first part of a series of two) some preliminary steps are made, namely, the diffraction problem is reduced to two matrix Riemann–Hilbert problems with exponential growth of unknown functions (for the symmetrical part and for the antisymmetrical part). For this, the Wiener–Hopf problems are formulated, they are reduced to auxiliary functional problems by applying the embedding formula, and finally the Riemann–Hilbert problems are formulated by applying the Hurd’s method.

In the second part the Riemann–Hilbert problems will be solved by a novel method of OE–equation.

1 Introduction

We study a 2D problem of diffraction by a segment bearing impedance boundary conditions on both sides. This problem can be considered as a cross-section of a 3D problem of diffraction by an infinitely long strip with finite width and zero thickness. The governing equation is the Helmholtz one, so the stationary problem is studied. No restriction is imposed on the relation between the wavelength and the width of the strip (length of the segment). The impedances of the sides are assumed to be equal.

The problem of diffraction by a segment has been studied extensively, but the vast majority of papers is related to the case of ideal (Dirichlet or Neumann) boundary conditions. A problem with ideal boundary conditions (ideal segment) admits an application of separation of variables method in the elliptical coordinates. As the result, the solution becomes expressed in terms of Matheu functions [1]. However this solution seems not attractive for applications and for analytical studies. Numerous attempts have been made to obtain a solution analogous to the Sommerfeld’s formula for the half-plane [2]. A review of these attempts can be found in [3]. Unfortunately, it has been found that the elegant approach of Riemann surface and Sommerfeld integral cannot be successfully used for the segment problem.

A good practical way to treat the segment problem at least in the short-wave approximation is the diffraction series approach. For an ideal segment this approach has been developed in [4, 5] and in many other papers.

Some mathematically important results for the ideal strip problem have been obtained in [6, 7, 8]. The problem of diffraction by an ideal strip was reduced there to the inverse monodromy problem for a confluent Heun's equation. Thus the problem of diffraction by an ideal strip has been solved at least in the mathematical sense. One of the authors contributed to this branch [9, 10, 11].

The problem of diffraction by an impedance segment seems much more complicated. In the case of high frequencies the method of diffraction series can be applied to this problem [12]. Otherwise one needs to solve an appropriate integral equation [13] numerically. Also there exist some hybrid techniques, which combine both analytical and numerical approach. By using such techniques computational time may be significantly reduced [14, 15, 16]. Besides, some approximate analytical methods, e. g. an approximate Wiener–Hopf technique [17] can be applied to this problem. Still the analytical theory of scattering by an impedance segment is far from being completed. Here we present some results that seem important and enable one to perform efficient calculations.

The first part of the paper describes the preliminary steps. Namely, the problem is formulated and symmetrized. After symmetrization, the symmetrical and the antisymmetrical problem are studied in parallel (they are slightly different). Following [18], for each of these two diffraction problems a functional problem is formulated. Then, auxiliary functional problems are formulated. The *embedding formula* expressing the directivity in terms of the auxiliary solutions is derived. This embedding formula is useful since it represents the directivity (which is a function of the angle of incidence and the angle of scattering) as a combination of functions depending on a single variable.

Method of embedding formula have been applied to many diffraction problems with different sets of auxiliary problems. In [6] embedding formula was derived for diffraction by an ideal strip. Problems with grazing incidence were taken to generate auxiliary solutions. In [19, 20, 21] embedding formula was obtained for diffraction by thin breakwaters using tricky manipulation with integral equations. Also embedding formula was derived for planar cracks in [22]. Edge Green's functions were used to generate auxiliary problems. In the current research we do not use this approach and just introduce auxiliary functional problems with a proper behaviour at infinity.

Then, following the procedure developed in [23] matrix Riemann–Hilbert problems are formulated for the auxiliary functional problems.

The second part of the paper will be dedicated to solving the matrix Riemann–Hilbert problems using a novel technique of the OE-equation.

2 Formulation of diffraction problem

Consider a 2D plane (x, y) . The scatterer is the segment $y = 0$, $-a < x < a$. Everywhere outside this segment the Helmholtz equation is valid:

$$\Delta u + k_0^2 u = 0 \quad (1)$$

where $u(x, y)$ is a field variable, and k_0 is a parameter. We assume that k_0 has a vanishing positive imaginary part in order to use the limiting absorption principle. The choice of time dependence is such that the wave traveling in the positive x -direction has the form $e^{ik_0 x}$.

The total field is a sum of the incident wave u^{in} and the scattered wave u^{sc} :

$$u = u^{\text{in}} + u^{\text{sc}},$$

where

$$u^{\text{in}} = \exp\{-ik_0(x \cos \theta^{\text{in}} + y \sin \theta^{\text{in}})\} \quad (2)$$

is a plane wave. Here θ^{in} is the angle of incidence; $0 \leq \theta^{\text{in}} \leq \pi/2$.

The total field should be one-side continuous on the scatterer and obey impedance boundary conditions on the faces of the scatterer:

$$\pm \frac{\partial u}{\partial y}(x, \pm 0) = \eta u(x, \pm 0), \quad -a < x < a. \quad (3)$$

Here η is the impedance parameter. Energy conservation or dissipation condition requires

$$\text{Im}[\eta] \leq 0. \quad (4)$$

The total field should obey Meixner's conditions near the vertices $(\pm a, 0)$. Namely, the integral of the "energy" combination $|\nabla u|^2 + |u|^2$ over any finite proximity of a vertex should be finite. Later on, the Meixner's condition will be reformulated as a restriction imposed on the growth of the field near the vertices.

The scattered field u^{sc} should also obey the Sommerfeld's radiation condition in the standard form:

$$\left(\frac{\partial u^{\text{sc}}}{\partial r} - ik_0 u^{\text{sc}} \right) = o(e^{ik_0 r} (k_0 r)^{-1/2}), \quad (5)$$

where $r = \sqrt{x^2 + y^2}$. Thus, the scattered field for large r can be written as follows:

$$u^{\text{sc}}(r, \theta) = \frac{\exp\{ik_0 r\}}{\sqrt{2\pi k_0 r}} S(\theta, \theta^{\text{in}}) + o(e^{ik_0 r} (k_0 r)^{-1/2}). \quad (6)$$

Here $\theta = \arctan(y/x)$, and $S(\theta, \theta^{\text{in}})$ is the *directivity* of the scattered field. This directivity should be found as the result of this research.

3 Symmetrization

Since the impedances of the faces of the scatterer are chosen to be equal, the problem can be split into the symmetrical and antisymmetrical parts:

$$u^{\text{sc}}(x, y) = u^{\text{a}}(x, y) + u^{\text{s}}(x, y), \quad (7)$$

where

$$u^{\text{a}}(x, y) = -u^{\text{a}}(x, -y), \quad u^{\text{s}}(x, y) = u^{\text{s}}(x, -y)$$

are the antisymmetrical and symmetrical parts, respectively.

The symmetrical and antisymmetrical parts correspond to the incident waves

$$u^{\text{in},\text{s}} = \frac{1}{2}[\exp\{-ik_0(x \cos \theta^{\text{in}} + y \sin \theta^{\text{in}})\} + \exp\{-ik_0(x \cos \theta^{\text{in}} - y \sin \theta^{\text{in}})\}],$$

$$u^{\text{in},\text{a}} = \frac{1}{2}[\exp\{-ik_0(x \cos \theta^{\text{in}} + y \sin \theta^{\text{in}})\} - \exp\{-ik_0(x \cos \theta^{\text{in}} - y \sin \theta^{\text{in}})\}],$$

respectively.

The problems for u^{a} and u^{s} can be formulated as mixed boundary value problems in the half-plane $y > 0$. Boundary conditions for u^{a} are as follows:

$$\left[\frac{\partial}{\partial y} - \eta \right] u^{\text{a}}(x, +0) = ik_0 \sin \theta^{\text{in}} \exp\{-ik_0 x \cos \theta^{\text{in}}\} \quad |x| < a, \quad (8)$$

$$u^{\text{a}}(x, 0) = 0, \quad |x| > a. \quad (9)$$

Boundary conditions for u^{s} are as follows:

$$\left[\frac{\partial}{\partial y} - \eta \right] u^{\text{s}}(x, +0) = \eta \exp\{-ik_0 x \cos \theta^{\text{in}}\} \quad |x| < a, \quad (10)$$

$$\frac{\partial}{\partial y} u^{\text{s}}(x, +0) = 0, \quad |x| > a. \quad (11)$$

Below we study the symmetrical and the antisymmetrical problem separately (in parallel). In both cases, we are interested in the field for $y > 0$ only.

The directivity of the scattered field is a sum of the symmetrical and antisymmetrical part:

$$S(\theta, \theta^{\text{in}}) = S^{\text{s}}(\theta, \theta^{\text{in}}) + S^{\text{a}}(\theta, \theta^{\text{in}}), \quad (12)$$

where the last two values are defined similarly to (6).

4 Local behavior of wave fields near the edges

Here we study the growth of the solutions near the vertices. This growth is limited by the Meixner's conditions.

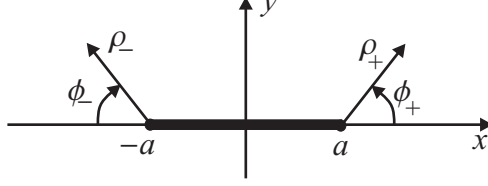


Fig. 1: Local coordinates

Introduce local cylindrical variables (ρ_{\pm}, ϕ_{\pm}) (Fig. 1). Consider the *total* field in the **antisymmetrical case**, i. e. consider the function $u = u^a + u^{\text{in},a}$. The Meixner's series for a solution has form

$$u(\rho, \phi) = \sum_m \sum_n (k_0 \rho)^{\nu_m} \log^n(k_0 \rho) f_{m,n}(\phi), \quad (13)$$

where $\rho = \rho_{\pm}$, $\phi = \phi_{\pm}$, $f_{m,n}(\phi) = f_{m,n}^{\pm}(\phi_{\pm})$. This series is substituted into the Helmholtz equation and into the boundary conditions. Also, some terms of the series are considered as prohibited according to the Meixner's condition mentioned above. As the result, we get the following asymptotic expansion of the field:

$$u = c(k_0 \rho)^{1/2} \sin(\phi/2) - \frac{2c\eta}{3\pi k_0} (k_0 \rho)^{3/2} \phi \cos(3\phi/2) - \frac{2c\eta}{3\pi k_0} (k_0 \rho)^{3/2} \log(k_0 \rho) \sin(3\phi/2) + O(\log^2(k_0 \rho) (k_0 \rho)^{5/2}). \quad (14)$$

Now consider the **symmetrical case**, i. e. let be $u = u^s + u^{\text{in},s}$. The asymptotics for this case is as follows:

$$u = d - \frac{\eta d}{\pi} \rho \log(k_0 \rho) \cos(\phi) + \frac{\eta d}{k_0 \pi} \rho \phi \sin(\phi) + O((k_0 \rho)^2 \log^2(k_0 \rho)). \quad (15)$$

Note that constants c and d in (14) and (15) are undetermined. Of course both constant take different values for two edges, i. e. totally we introduce four constants c_{\pm} and d_{\pm} here.

5 Formulation of Wiener–Hopf functional problems

5.1 Antisymmetrical case

Consider domain Ω shown in Fig. 2. This domain is bounded by a part of x -axis, two small arcs (having radii $\epsilon \rightarrow 0$) encircling the vertices, and a large arc (having radius $R \rightarrow \infty$) mimicking the infinity. Consider two functions, both solutions of Helmholtz equation (1) in Ω . The first function is u^a (the scattered

field in the antisymmetrical case), and the second function is an outgoing or decaying plane wave w :

$$w = w(k, x, y) = \exp \{i (kx + \xi(k)y)\}, \quad (16)$$

$$\xi(k) \equiv \sqrt{k_0^2 - k^2}, \quad (17)$$

where k is a real value. The branch of square root ξ is chosen in such a way that while $|k| < \text{Re}[k_0]$ the values of the square root are close to positive real. By continuity, the values of the square root for $|k| > \text{Re}[k_0]$ are close to positive imaginary (the real axis passes below the point k_0 due to the limiting absorption principle). Note that w is a solution of the Helmholtz equation for each value of parameter k .

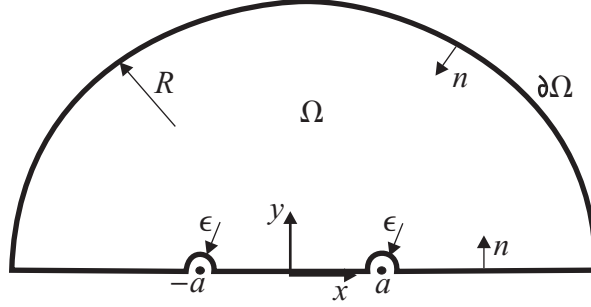


Fig. 2: Contour for the Green's formula

Apply the Green's formula to these two functions in Ω :

$$\int_{\partial\Omega} \left[\frac{\partial u^a}{\partial n} w - \frac{\partial w}{\partial n} u^a \right] dl = 0. \quad (18)$$

Since function u^a obeys the radiation condition, the integral over the large arc tends to zero as $R \rightarrow \infty$. The integrals over small arcs tend to zero as $\epsilon \rightarrow 0$ due to the local asymptotic expansions at the vertices. Thus, only the integral over the parts of the x -axis should be considered.

Define the following values:

$$\check{U}_-(k) = \int_{-\infty}^{-a} \left[\frac{\partial u^a}{\partial n} w - \frac{\partial w}{\partial n} u^a \right] dx = \int_{-\infty}^{-a} \frac{\partial u^a(x, +0)}{\partial y} e^{ikx} dx, \quad (19)$$

$$\check{U}_0(k) = \int_{-a}^a \left[\frac{\partial u^a(x, +0)}{\partial n} w(x, +0) - \frac{\partial w(x, +0)}{\partial n} u^a(x, +0) \right] dx, \quad (20)$$

$$\check{U}_+(k) = \int_a^{\infty} \left[\frac{\partial u^a}{\partial n} w - \frac{\partial w}{\partial n} u^a \right] dx = \int_a^{\infty} \frac{\partial u^a(x, +0)}{\partial y} e^{ikx} dx. \quad (21)$$

According to (18) the following functional equations are valid for all real k :

$$\check{U}_-(k) + \check{U}_0(k) + \check{U}_+(k) = 0. \quad (22)$$

Expression (20) can be transformed using (8):

$$\begin{aligned} \check{U}_0(k) &= (\eta - i\xi(k)) \int_{-a}^a u^a(x, +0) e^{ikx} dx + \\ &\frac{k_0 \sin \theta^{\text{in}}}{k - k_*} (\exp\{i(k - k_*)a\} - \exp\{-i(k - k_*)a\}), \end{aligned} \quad (23)$$

where

$$k_* = k_0 \cos \theta^{\text{in}}.$$

Define the values

$$U_-(k) \equiv \check{U}_-(k) - \frac{k_0 \sin \theta^{\text{in}}}{k - k_*} \exp\{-i(k - k_*)a\} \quad (24)$$

$$U_0(k) \equiv (\eta - i\xi(k)) \int_{-a}^a u^a(x, +0) e^{ikx} dx \quad (25)$$

$$U_+(k) \equiv \check{U}_+(k) + \frac{k_0 \sin \theta^{\text{in}}}{k - k_*} \exp\{i(k - k_*)a\}. \quad (26)$$

According to (22) these values obey the functional equation

$$U_-(k) + U_0(k) + U_+(k) = 0. \quad (27)$$

Functions \check{U}_j , $j = -, 0, +$ are defined as Fourier transforms taken on some parts of the real axis. Thus, standard theorems can be used to establish properties of these functions as well as the properties of U_j :

Property 1 Function $U_-(k)$ defined by (24) and (19) can be analytically continued onto the whole lower half-plane from the real axis, and it is regular there. Note that since we assume that k_0 has a negligibly small positive imaginary part, the important point $k = -k_0$ belongs to the lower half-plane, and the function $U_-(k)$ is regular at this point.

Property 2 Similarly, function $U_+(k)$ defined by (26) and (21) can be analytically continued onto the whole upper half-plane including $k = k_0$, and it is regular everywhere in the upper half-plane except a pole at $k = k_*$. At this pole function U_+ has a prescribed residue equal to $k_0 \sin \theta^{\text{in}}$.

Property 3 Function

$$\tilde{U}_0(k) = (\eta - i\xi(k))^{-1} U_0(k) \quad (28)$$

is regular on the whole complex plane k .

Property 4 Applying Watson's lemma to the integral representations (19), (20), (21) we can get the following growth estimations as $|k| \rightarrow \infty$ in the domains of *a priori* regularity of the unknown functions:

$$U_+(k) = O(k^{-1/2} e^{ika}), \quad \text{Arg}[e^{-i\pi/2} k] \leq \pi/2, \quad (29)$$

$$U_-(k) = O(k^{-1/2} e^{-ika}), \quad \text{Arg}[e^{i\pi/2} k] \leq \pi/2, \quad (30)$$

$$U_0(k) = O(k^{-1/2} e^{-ika}), \quad \text{Arg}[e^{-i\pi/2} k] \leq \pi/2, \quad (31)$$

$$U_0(k) = O(k^{-1/2} e^{ika}), \quad \text{Arg}[e^{i\pi/2} k] \leq \pi/2, \quad (32)$$

Note that estimations (29), (30) require some algebra to derive.

Introduce cuts \mathcal{G}_1 and \mathcal{G}_2 going from $-k_0$ and k_0 to infinity (see Fig. 3). These cuts go along the lines corresponding to the values of the square root $\pm\sqrt{k_0^2 - k^2}$ taken for real k . Function U_- can be naturally continued to the

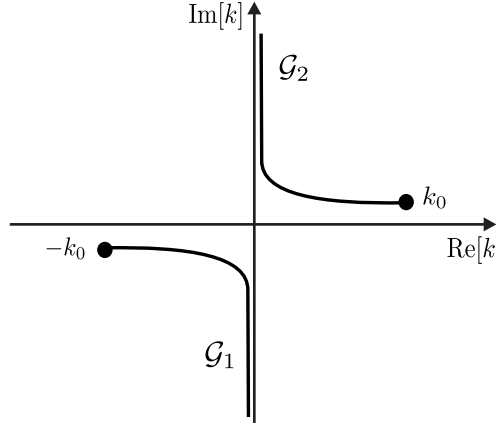


Fig. 3: Cuts \mathcal{G}_1 and \mathcal{G}_2

lower half-plane, function U_+ can be naturally continued to the upper half-plane, and function U_0 can be continued to the whole plane with the cuts \mathcal{G}_1 and \mathcal{G}_2 . However, using relations

$$U_-(k) = -U_0(k) - U_+(k), \quad U_+(k) = -U_0(k) - U_-(k)$$

the functions U_- can be continued to the upper half-plane with a cut \mathcal{G}_2 , and the functions U_+ can be continued to the lower half-plane with a cut \mathcal{G}_1 . Moreover, it is possible to study the Riemann surface of each function from the set (U_-, U_+, U_0) , and prove that all branch points have order two and affixes $\pm k_0$.

These properties enable us to formulate a functional problem for the functions U_{\pm} :

Problem 1 Find functions $U_+(k)$, $U_-(k)$, regular in the complex plane with the cuts \mathcal{G}_1 and \mathcal{G}_2 , such that

- function U_- is regular in the lower half-plane;
- function U_+ is regular in the upper half-plane except a simple pole at $k = k_*$ with a residue equal to $k_0 \sin \theta^{\text{in}}$;
- function $(\eta - i\xi(k))^{-1}U_0(k)$ is regular on the whole plane (here U_0 is defined as $U_0 \equiv -(U_+ + U_-)$);
- functions U_+ , U_- , \tilde{U}_0 obey growth restrictions (29), (30), (31), (32).

The formulation of the functional problem means that we forget about the definition of the unknown functions through the wave fields, and look for functions $U_+(k)$, $U_-(k)$ obeying Problem 1 and having arbitrary nature.

Let a solution of the functional problem be found. Let us describe the link between the directivity $S^a(\theta)$ for the antisymmetrical problem and the solution of the functional problem. Apply Green's formula (18) to the domain Ω , take u^a as u , and $u^{\text{in},a}(x, y)$ as w . The integral over the large arc tends to a constant linked with the directivity. The result is as follows:

$$S^a(\theta, \theta^{\text{in}}) = -e^{-i\pi/4} k_0 \sin \theta \tilde{U}_0(-k_0 \cos(\theta)). \quad (33)$$

Note that \tilde{U}_0 depends on θ^{in} implicitly.

5.2 Functional problem for the symmetrical case

In the symmetrical case define functions $V_-(k)$, $V_+(k)$, $V_0(k)$ by formulae

$$V_-(k) = \int_{-\infty}^{-a} \exp\{ikx\} u^s(x, +0) dx - \frac{i}{k - k_*} \exp\{-i(k - k_*)a\}, \quad (34)$$

$$V_0(k) = \frac{i(\eta - i\xi(k))}{\eta\xi(k)} \int_{-a}^a \exp\{ikx\} \frac{\partial u^s(x, +0)}{\partial y} dx, \quad (35)$$

$$V_+(k) = \int_a^{\infty} \exp\{ikx\} u^s(x, +0) dx + \frac{i}{k - k_*} \exp\{i(k - k_*)a\}, \quad (36)$$

which are similar to (24), (25), (26). A functional equation is valid for these functions:

$$V_-(k) + V_0(k) + V_+(k) = 0. \quad (37)$$

The growth estimations for the new unknown functions are as follows:

$$V_+(k) = O(k^{-1} e^{ika}), \quad \text{Arg}[e^{-i\pi/2} k] \leq \pi/2, \quad (38)$$

$$V_-(k) = O(k^{-1} e^{-ika}), \quad \text{Arg}[e^{i\pi/2} k] \leq \pi/2, \quad (39)$$

$$V_0(k) = O(k^{-1} e^{-ika}), \quad \text{Arg}[e^{-i\pi/2} k] \leq \pi/2, \quad (40)$$

$$V_0(k) = O(k^{-1} e^{ika}), \quad \text{Arg}[e^{i\pi/2} k] \leq \pi/2. \quad (41)$$

The functional problem for the functions V_{\pm} is as follows:

Problem 2 Find functions $V_+(k)$, $V_-(k)$, regular in the complex plane with the cuts \mathcal{G}_1 and \mathcal{G}_2 , such that

- function $V_-(k)$ is regular in the lower half-plane;
- function $V_+(k)$ is regular in the upper half-plane except a simple pole at $k = k_*$ with residue equal to i ;
- function

$$\tilde{V}_0 = \frac{\eta\xi(k)}{i(\eta - i\xi(k))} V_0(k) \quad (42)$$

is regular in the whole plane (here V_0 is defined as $V_0 \equiv -(V_+ + V_-)$);

- functions V_+ , V_- , \tilde{V}_0 obey growth restrictions (38), (39), (40), (41).

The expression for the directivity of the symmetrical problem is as follows:

$$S^s(\theta, \theta^{\text{in}}) = e^{-i\pi/4} \tilde{V}_0(-k_0 \cos(\theta)). \quad (43)$$

6 Auxiliary Wiener–Hopf functional problem and embedding formula

6.1 Auxiliary functions. Antisymmetrical problem

Consider Problem 1. Here we modify this functional problem and formulate a problem for the auxiliary functions. The following modifications are made. First, two pairs of auxiliary functions are introduced. They are (U_-^1, U_+^1) , (U_-^2, U_+^2) . This enables us to construct a basis of solutions for a family of initial functional problems indexed by parameter θ^{in} . Second, functions $U_+^{1,2}$ are required to have no poles (i. e. the conditions of analyticity become more strict). Third, faster growth at infinity is allowed (i. e. growth restriction become weaker).

Problem 3 Find functions $U_+^{1,2}(k)$, $U_-^{1,2}(k)$, regular in the complex plane with the cuts \mathcal{G}_1 and \mathcal{G}_2 , such that

- functions $U_-^{1,2}$ are regular in the lower half-plane;
- functions $U_+^{1,2}$ are regular in the upper half-plane;
- functions

$$\tilde{U}_0 = (\eta - i\xi(k))^{-1} U_0^{1,2}(k) \quad (44)$$

are regular on the whole plane (here functions $U_0^{1,2}$ are defined as $U_0^{1,2} \equiv -(U_+^{1,2} + U_-^{1,2})$);

- functions U_+ , U_- , \tilde{U}_0 obey growth restrictions (45), (46), (47), (48) formulated below.

The growth restrictions for this functional problem have the following form:

$$U_+^j(k) = \delta_{j,2}(e^{-i\pi/2}k)^{1/2}e^{ika} + O(k^{-1/2}e^{ika}), \quad \text{Arg}[e^{-i\pi/2}k] \leq \pi/2, \quad (45)$$

$$U_-^j(k) = \delta_{j,1}(e^{i\pi/2}k)^{1/2}e^{-ika} + O(k^{-1/2}e^{-ika}), \quad \text{Arg}[e^{i\pi/2}k] \leq \pi/2, \quad (46)$$

$$\tilde{U}_0^j(k) = -\delta_{j,1}(e^{-i\pi/2}k)^{-1/2}e^{-ika} + O(k^{-3/2}e^{-ika}), \quad \text{Arg}[e^{-i\pi/2}k] \leq \pi/2, \quad (47)$$

$$\tilde{U}_0^j(k) = -\delta_{j,2}(e^{i\pi/2}k)^{-1/2}e^{ika} + O(k^{-3/2}e^{ika}), \quad \text{Arg}[e^{i\pi/2}k] \leq \pi/2, \quad (48)$$

where $j = 1, 2$, and δ is the Kronecker's symbol.

Organize the solution of the auxiliary functional problem as a matrix

$$U(k) = \begin{pmatrix} U_-^1(k) & U_+^1(k) \\ U_-^2(k) & U_+^2(k) \end{pmatrix}. \quad (49)$$

Let us show that the solution of Problem 3 is unique. Namely, let there exist two such solutions U and \bar{U} . Consider the expression $J = \bar{U}U^{-1}$. This expression is equal to

$$J = \frac{1}{D} \begin{pmatrix} D_{1,1} & D_{1,2} \\ D_{2,1} & D_{2,2} \end{pmatrix} \quad (50)$$

where

$$\begin{aligned} D &= |U|, \\ D_{1,1} &= \begin{vmatrix} \bar{U}_-^1(k) & \bar{U}_+^1(k) \\ U_-^2(k) & U_+^2(k) \end{vmatrix}, \\ D_{1,2} &= \begin{vmatrix} U_-^1(k) & U_+^1(k) \\ \bar{U}_-^1(k) & \bar{U}_+^1(k) \end{vmatrix}, \\ D_{2,1} &= \begin{vmatrix} \bar{U}_-^2(k) & \bar{U}_+^2(k) \\ U_-^2(k) & U_+^2(k) \end{vmatrix}, \\ D_{2,2} &= \begin{vmatrix} U_-^1(k) & U_+^1(k) \\ \bar{U}_-^2(k) & \bar{U}_+^2(k) \end{vmatrix}, \end{aligned}$$

where $|\cdot|$ denotes determinant of the matrix.

All five determinants can be analyzed as follows. Consider D as an example. Study two representations of this determinant (they are equivalent due to linear dependence of U_-^j , U_+^j , and \tilde{U}_0^j):

$$D = -(\eta - i\xi(k)) \begin{pmatrix} U_-^1 & \tilde{U}_0^1 \\ U_-^2 & \tilde{U}_0^2 \end{pmatrix} = -(\eta - i\xi(k)) \begin{pmatrix} \tilde{U}_0^1 & U_+^1 \\ \tilde{U}_0^2 & U_+^2 \end{pmatrix}. \quad (51)$$

The first representation can be used to study the behaviour of

$$\tilde{D}(k) \equiv -(\eta - i\xi(k))^{-1}D(k)$$

in the lower half-plane, and the second representation can be used to study the behaviour of the same function in the upper half-plane. One can see that \tilde{D}

is analytical in both half-planes, and grows as a constant equal to -1 in both half-planes. Thus, according to Liouville's theorem,

$$\tilde{D} \equiv -1.$$

A similar reasoning can be applied to each of four other determinants. The result is

$$J(k) \equiv I,$$

which is the identity matrix, i. e. the solution is unique. Note that the determinant $D(k)$ can have no zeros except the zeros of the function $\eta - i\xi(k)$.

6.2 Auxiliary functions. Symmetrical problem

Similarly to the antisymmetrical case, introduce an auxiliary functional problem for the symmetrical case.

Problem 4 Find functions $V_+^1(k)$, $V_+^2(k)$, $V_-^1(k)$, $V_-^2(k)$, regular in the complex plane with the cuts \mathcal{G}_1 and \mathcal{G}_2 , such that

- functions V_-^j are regular in the lower half-plane;
- functions V_+^j are regular in the upper half-plane;
- functions

$$\tilde{V}_0^j \equiv -\frac{\xi(k)}{i(\eta - i\xi(k))}(V_-^j + V_+^j) \quad (52)$$

are regular on the whole plane;

- functions V_+^j , V_-^j , \tilde{V}_0^j obey growth restrictions (53), (54), (55), (56) formulated below.

The growth conditions for this functional problem have the following form:

$$V_+^j(k) = \delta_{j,2}e^{ika} + O(k^{-1}e^{ika}), \quad \text{Arg}[e^{-i\pi/2}k] \leq \pi/2, \quad (53)$$

$$V_-^j(k) = \delta_{j,1}e^{-ika} + O(k^{-1}e^{-ika}), \quad \text{Arg}[e^{i\pi/2}k] \leq \pi/2, \quad (54)$$

$$\tilde{V}_0^j(k) = -\delta_{j,1}e^{-ika} + O(k^{-1}e^{-ika}), \quad \text{Arg}[e^{-i\pi/2}k] \leq \pi/2, \quad (55)$$

$$\tilde{V}_0^j(k) = -\delta_{j,2}e^{ika} + O(k^{-1}e^{ika}), \quad \text{Arg}[e^{i\pi/2}k] \leq \pi/2. \quad (56)$$

The solution of the functional problem can be organized as a matrix

$$V(k) = \begin{pmatrix} V_-^1(k) & V_+^1(k) \\ V_-^2(k) & V_+^2(k) \end{pmatrix}. \quad (57)$$

Using representation similar to (50) one can show that Problem 4 has a unique solution.

6.3 Embedding formula

Consider the **antisymmetrical** case. Let row vector (U_-, U_+) be a solution of Problem 1, and let $U(k)$ be a solution of Problem 3 in the matrix form (49). Find functions $r_1(k)$ and $r_2(k)$ such that

$$(U_-(k), U_+(k)) = (r_1(k), r_2(k)) \begin{pmatrix} U_-^1(k) & U_+^1(k) \\ U_-^2(k) & U_+^2(k) \end{pmatrix}. \quad (58)$$

Due to Cramer's rule,

$$r_1 = \frac{D_1}{D}, \quad r_2 = \frac{D_2}{D}, \quad (59)$$

where

$$D_1 = \begin{vmatrix} U_-(k) & U_+(k) \\ U_-^2(k) & U_+^2(k) \end{vmatrix}, \quad D_2 = \begin{vmatrix} U_-^1(k) & U_+^2(k) \\ U_-(k) & U_+(k) \end{vmatrix}. \quad (60)$$

Determinant D was calculated in the previous section using representation (51). Determinants D_1, D_2 can be analyzed similarly to determinant D , namely there exist two representations for each determinant enabling one to study these determinants in the upper and lower half-plane:

$$D_1 = -(\eta - i\xi(k)) \begin{pmatrix} U_- & \tilde{U}_0 \\ U_-^2 & \tilde{U}_0^2 \end{pmatrix} = -(\eta - i\xi(k)) \begin{pmatrix} \tilde{U}_0 & U_+ \\ \tilde{U}_0^2 & U_+^2 \end{pmatrix}, \quad (61)$$

$$D_2 = -(\eta - i\xi(k)) \begin{pmatrix} U_-^1 & \tilde{U}_0^1 \\ U_- & \tilde{U}_0 \end{pmatrix} = -(\eta - i\xi(k)) \begin{pmatrix} \tilde{U}_0^1 & U_+^1 \\ \tilde{U}_0 & U_+ \end{pmatrix}. \quad (62)$$

Using these representations and applying the Liouville's theorem one can prove that

$$D_1 = \frac{(\eta - i\sqrt{k_0^2 - k^2})}{k - k_*} R_1, \quad (63)$$

$$D_2 = \frac{(\eta - i\sqrt{k_0^2 - k^2})}{k - k_*} R_2, \quad (64)$$

where R_1, R_2 are some constants. R_1, R_2 can be obtained by calculating residues of determinants D_1, D_2 at the point $k = k_*$. These residues can be found either from (61), (62) or from (63), (64). Comparing these representations, obtain

$$R_1 = -\sqrt{k_0^2 - k_*^2} \tilde{U}_0^2(k_*), \quad R_2 = \sqrt{k_0^2 - k_*^2} \tilde{U}_0^1(k_*). \quad (65)$$

Substituting r_1 and r_2 into (60) obtain the embedding formula:

$$\tilde{U}_0(k, k_*) = \frac{\xi(k_*)}{k - k_*} \left(\tilde{U}_0^1(k_*) \tilde{U}_0^2(k) - \tilde{U}_0^1(k) \tilde{U}_0^2(k_*) \right). \quad (66)$$

According to embedding formula we can focus our efforts on finding the solution of Problem 3, namely on functions $U_0^j(k)$, $j = 1, 2$.

Conducting a similar procedure one can obtain an embedding formula for the **symmetrical case**:

$$\tilde{V}_0(k, k_*) = \frac{i\eta}{(k - k_*)} \left(\tilde{V}_0^2(k_*) \tilde{V}_0^1(k) - \tilde{V}_0^2(k) \tilde{V}_0^1(k_*) \right). \quad (67)$$

7 Matrix Riemann–Hilbert formulation for auxiliary functional problems

7.1 Antisymmetrical problem

Here we present a matrix Riemann–Hilbert formulation for the antisymmetrical case.

Let us make some preliminary steps. Consider the cuts \mathcal{G}_1 and \mathcal{G}_2 (see Fig. 4, left). The values on the left shores (when going from $\pm k_0$ to ∞) of the cuts are denoted by symbols with lower index L ; the values on the right shores are denoted by index R .

Consider the bypasses about $\pm k_0$ and going from a point on the left shore to the right shore, i. e. going in the positive direction. Our current aim is to describe the transformation of the matrix U occurring as a result of the bypass. Namely, let us prove that

$$U_R(k) = U_L(k) M_1(k), \quad k \in \mathcal{G}_1, \quad (68)$$

$$U_R(k) = U_L(k) M_2(k), \quad k \in \mathcal{G}_2, \quad (69)$$

with

$$M_1(k) = \begin{pmatrix} 1 & 2i\xi/(\eta - i\xi) \\ 0 & (\eta + i\xi)/(\eta - i\xi) \end{pmatrix}, \quad (70)$$

$$M_2(k) = \begin{pmatrix} (\eta + i\xi)/(\eta - i\xi) & 0 \\ 2i\xi/(\eta - i\xi) & 1 \end{pmatrix}. \quad (71)$$

The analytic continuation of the square root $\xi(k) \equiv \sqrt{k_0^2 - k^2}$ on the cuts $\mathcal{G}_{1,2}$ is defined as follows. This square root is equal to k_0 for $k = 0$. Then, introduce the paths shown in Fig. 4 (right). These paths go from zero to the left shores of $\mathcal{G}_{1,2}$. The values of the square root on $\mathcal{G}_{1,2}$ is taken as the result of the continuation along these paths. The values of the square root are taken for $M_{1,2}$ from the left shores.

Derive (69). Consider contour \mathcal{G}_2 associated with matrix M_2 . Continue functional equation (22):

$$(U_-^j(k))_L = -U_+^j(k) - (\eta - i\xi(k)) \tilde{U}_0^j(k), \quad (72)$$

$$(U_-^j(k))_R = -U_+^j(k) - (\eta + i\xi(k)) \tilde{U}_0^j(k). \quad (73)$$

Then,

$$(U_-^j(k))_R = \frac{\eta + i\xi(k)}{\eta - i\xi(k)} (U_-^j(k))_L + \frac{2i\xi(k)}{\eta - i\xi(k)} U_+^j(k).$$

Note that functions U_+^j and \tilde{U}_0^j are not labeled as R or L , since they do not change their values after the considered bypass. Thus, relations (69) and (71) are valid. Similarly one can prove (68) and (70).

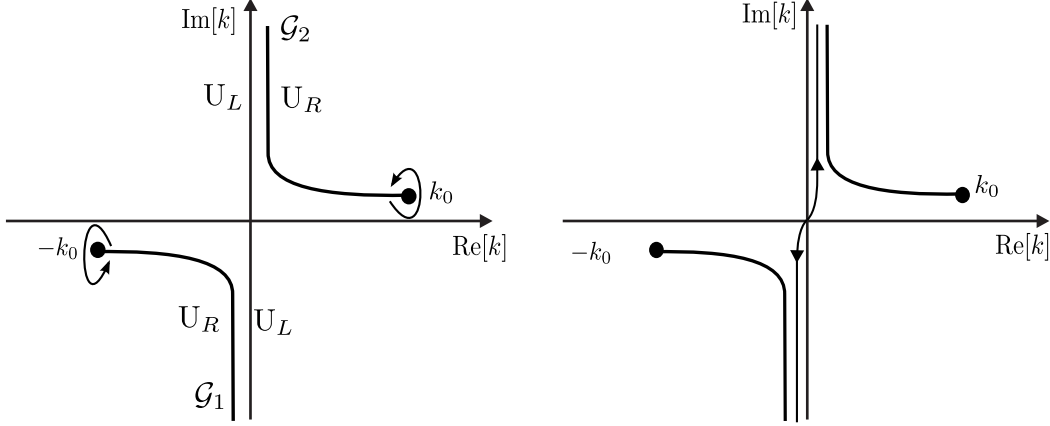


Fig. 4: (left) Bypasses around k_0 and $-k_0$. (right) Analytical continuation of the square roots

Reformulate the growth restrictions (47) and (48) according to (22) as follows:

$$U_-^j = i \delta_{j,1} (e^{-i\pi/2} k)^{1/2} e^{-ika} + O(k^{-1/2} e^{-ika}), \quad \text{Arg}[e^{-i\pi/2} k] \leq \pi/2, \quad (74)$$

$$U_+^j = i \delta_{j,2} (e^{i\pi/2} k)^{1/2} e^{ika} + O(k^{-1/2} e^{ika}), \quad \text{Arg}[e^{i\pi/2} k] \leq \pi/2. \quad (75)$$

Both restrictions are related to the continuations along the paths shown in Fig. 4.

Now we can formulate a Riemann–Hilbert problem for U :

Problem 5 Find a matrix function $U(k)$ of elements (49) such that

- it is regular on the plane cut along the lines $\mathcal{G}_{1,2}$;
- it obeys functional equations (68), (69) with coefficients (71), (70) on the cuts;
- it obeys growth restrictions (45), (46), (74), (75);
- functions $U_+^j(k) + U_-^j(k)$, $j = 1, 2$ have zeros at $k = k' \equiv \sqrt{k_0^2 + \eta^2}$;
- functions U_\pm^j grow no faster than a constant near the points $\pm k_0$.

The fourth condition (concerning zeros at $\pm k'$) are difficult to take into account, so we would like to eliminate it. Consider Riemann surface of the function $\sqrt{k_0^2 - k^2}$ cut along the lines $\mathcal{G}_{1,2}$. The surface is split into two sheets by the cuts. The sheet to which the point $\sqrt{k_0^2 - 0^2} = k_0$ belongs will be called the physical sheet. Consider the function $\eta - i\sqrt{k_0^2 - k^2}$ on this surface. Note that this function has two zeros only on one sheet (on the physical one or on the other one). If the zeros belong to the physical sheet, deform the contours $\mathcal{G}_{1,2}$ such that:

- the end points remain the same;
- contour \mathcal{G}_2 remains symmetrical to \mathcal{G}_1 with respect to zero;
- zeros of $\eta - i\sqrt{k_0^2 - k^2}$ finally become not belonging to the physical sheet.

A scheme of such contour deformation is shown in Fig. 5.

If the zeros do not belong to the physical sheet from the very beginning, then no deformation is needed. The domain of η for which the zeros of $\eta - i\sqrt{k_0^2 - k^2}$ belong to the physical sheet (and the deformation is needed) is

$$\text{Im}[\eta] < 0, \quad \text{Re}[\eta] < 0, \quad (76)$$

i. e. it is the third quadrant of the complex plane.

Denote the resulting contours (deformed if the deformation is needed or undeformed otherwise) by $\mathcal{G}'_{1,2}$.

Remark. Positions of the points k' on the Riemann surface of $\sqrt{k_0^2 - k^2}$ can be found from condition (4). Namely, the boundary between the allowed values of η and prohibited values is the real axis. Consider the function $k' = k'(\eta)$. This function maps the real axis of η into the parts $\mathcal{G}_1'' = (-\infty, -k_0)$, $\mathcal{G}_2'' = (k_0, \infty)$ of the real axis. Consider the Riemann surface of $\sqrt{k_0^2 - k^2}$ cut along $\mathcal{G}_{1,2}$. The surface will be split into two sheets. Again, call the sheet containing the point $\sqrt{k_0^2 - 0^2} = k_0$ the physical sheet. The boundary $\text{Im}[\eta] = 0$ corresponds to the cuts $\mathcal{G}_{1,2}$. The area $\text{Im}[\eta] < 0$ corresponds to the unphysical sheet.

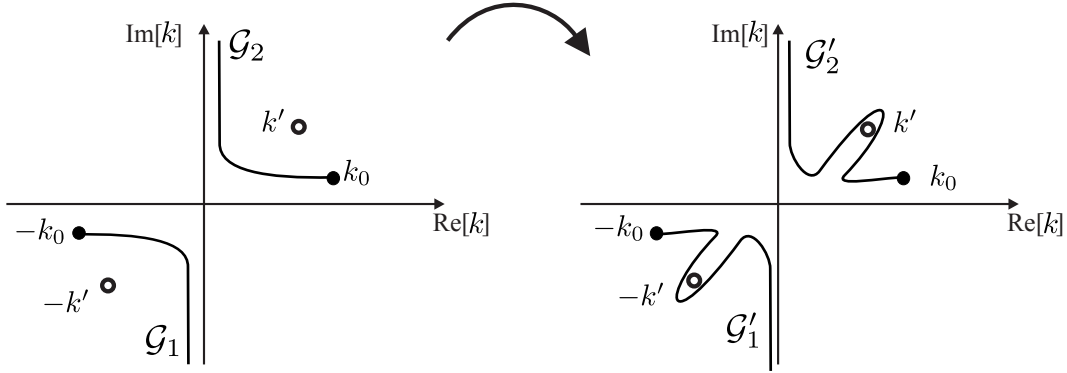


Fig. 5: Deformation of the cuts $\mathcal{G}_{1,2}$

Formulate the functional problem for the contours $\mathcal{G}'_{1,2}$. According to the principles of analytical continuation, relations (68), (69) remain valid with the same matrices (71), (70). Thus, the formulation of the problem is almost the same:

Problem 6 Find a matrix function $U(k)$ of elements (49) such that

- it is regular on the plane cut along the lines $\mathcal{G}'_{1,2}$;
- it obeys functional equations (68), (69) with coefficients (71), (70) on the cuts;
- it obeys growth restrictions (45), (46), (74), (75);
- functions U_{\pm}^j grow no faster than a constant near the points $\pm k_0$.

7.2 Symmetrical problem

Similarly to the antisymmetrical case, there are two functional equations describing the transformation of unknown functions at the cuts:

$$V_R(k) = V_L N_1(k), \quad k \in \mathcal{G}_1, \quad (77)$$

$$V_R(k) = V_L N_2(k), \quad k \in \mathcal{G}_2. \quad (78)$$

$$N_1(k) = \begin{pmatrix} 1 & -2\eta/(\eta - i\zeta) \\ 0 & (\eta + i\zeta)/(i\zeta - \eta) \end{pmatrix}, \quad (79)$$

$$N_2(k) = \begin{pmatrix} (\eta + i\zeta)/(i\zeta - \eta) & 0 \\ -2\eta/(\eta - i\zeta) & 1 \end{pmatrix}, \quad (80)$$

Reformulate growth restrictions (55), (56) according to (52) as follows:

$$V_-^j = \delta_{j,1} e^{-ika} + O(k^{-1} \log(k) e^{-ika}), \quad \text{Arg}[e^{-i\pi/2} k] \leq \pi/2, \quad (81)$$

$$V_+^j = \delta_{j,2} e^{ika} + O(k^{-1} \log(k) e^{ika}), \quad \text{Arg}[e^{i\pi/2} k] \leq \pi/2. \quad (82)$$

Finally, formulate a functional problem for V .

Problem 7 Find a matrix function $V(k)$ of elements (57) such that

- it is regular on the plane cut along the lines $\mathcal{G}_{1,2}$;
- it obeys functional equations (77), (78) with coefficients (79), (80) on the cuts;
- it obeys growth restrictions (53), (54), (81), (82);
- functions V_{\pm}^j grow no faster than $(\sqrt{k_0 \mp k})^{-1/2}$ near the points $\pm k_0$.

8 Conclusion

The problem of diffraction by impedance strip is symmetrized and reduced to two Wiener–Hopf functional problems (Problem 1 and 2) leading to directivities $S^a(\theta, \theta^{\text{in}})$ and $S^s(\theta, \theta^{\text{in}})$. Then auxiliary functional problems (Problem 3 and 4) are introduced. Using embedding formulae (66) and (67) a simple connection with Problem 1 and 2 is established. Riemann–Hilbert problems (Problem 6 and 7) for auxiliary solutions are formulated.

In the second part of the paper the family of Riemann–Hilbert problems indexed by an artificial parameter will be introduced. A differential equation will be built with respect to this parameter. A novel technique of OE–equation will be applied to solve this equation and find the solution of original problem. Some numerical results will be presented.

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